A CLASS OF CONTROLLABLE NONLINEAR SYSTEMS[*]

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Abstract

Sufficient conditions for global and local controllability of the nonlinear system

$$\frac{d}{dt} x(t) = A(t,x(t),u(t))x(t) + B(t)u(t) + f(t,x(t),u(t))$$

and its perturbed system are given. These conditions extend some previous results through the removal of certain boundedness conditions involving the functions (A,f) and their partial derivatives.

I. INTRODUCTION

The purpose of this correspondence is to extend the results of [1] by considering a more general class of nonlinear control systems. The results obtained in this work provide sufficient conditions for global and local controllability of perturbed nonlinear systems.

Consider the nonlinear time-varying system

$$\frac{d}{dt} x(t) = A(t,x(t),u(t))x(t) + B(t)u(t) + f(t,x(t),u(t)) \tag{1}$$

$t\epsilon[t_o,t_1]$, where $x(t)$ is a nx1 state vector; $u(t)$ is a mx1 input vector; A,B,f are nxn, nxm, nx1 matrix-valued functions, respectively.

Denote $C_{nm}[t_o,t_1]$ as the Banach space of continuous $R^n \times R^m$ valued functions on $[t_o,t_1]$ with the uniform norm $||[x(t),u(t)]|| = |x(t)| + |u(t)|$, where $|w(t)| = \max\limits_{i} \max\limits_{t\epsilon[t_o,t_1]} |w_i(t)|$ and $|w_i(t)|$ is the absolute value of $w_i(t)$, the element of $w(t)$.

Define the norm of a continuous nxm matrix-valued function $S(t)$ by $||S(t)|| = \max\limits_{i} \sum\limits_{j=1}^{m} \max\limits_{t\epsilon[t_o,t_1]} |S_{ij}(t)|$, where S_{ij} are elements of S.

Given (x_o,x_1) as the initial and final state, respectively of (1). The problem is to find a continuous input function $u(t)$, defined in $[t_o,t_1]$, which steers system (1) from x_o at t_o to x_1 at t_1. The usual definitions of globally and locally completely and totally controllable are assumed [2].

For each fixed element $[z,v] \epsilon C_{nm} [t_o,t_1]$, consider the following system

$$\frac{d}{dt} x(t) = A(t,z(t),v(t))x(t) + B(t)u(t) + f(t,z(t),v(t)) \tag{2}$$

The solution of the system (2) with $x(t_o) = x_o$ is given by

$$x(t) = \Phi(t,t_o;z,v)x_o + \int_{t_o}^{t} \Phi(t,s;z,v)B(s)u(s)ds$$

$$+ \int_{t_o}^{t} \Phi(t,s;z,v)f(s,z,v)ds \tag{3}$$

where $\Phi(t,t_o;z,v)$ is the state transition matrix of the homogeneous system
$\frac{d}{dt} x(t) = A(t,z(t),v(t))x(t)$, $\Phi(t_o,t_o;z,v) = I$ the identity matrix. Define
the controllability matrix by

$$G(t_o,t;z,v) = \int_{t_o}^{t} \Phi(t,s;z,v)B(s)B'(s) \Phi'(t,s;z,v)ds \tag{4}$$

the prime denotes the matrix transpose operation. Obviously, $G(t_o,t;z,v)$ is
symmetric and non-negative definite.

II. GLOBAL CONTROLLABILITY RESULT

Theorem 1: The system (1) is globally (a) completely controllable at t_o,
or (b) totally controllable if the following conditions are satisfied.

(i) $B(t)$ has a continuous first derivative with respect to t,

(ii) $A(t,x,u)$, $A_x(t,x,u)$, $A_u(t,x,u)$, $f(t,x,u)$, $f_x(t,x,u)$, and $f_u(t,x,u)$ are
continuous and bounded in $[t_o,t_1] \times R^n \times R^m$,

(iii) there exists a positive constant q such that

$$\inf_{[z,v] \in C_{nm}} \det_{[t_o,t_1]} G(t_o,t_1;z,v) \geq q \tag{5}$$

(a) for some $t_1 > t_o$, or (b) for all t_o and all $t_1 > t_o$.

Proof: The proof of the theorem is based on the Schauder's fixed point
theorem of the following version, "Every continuous map which maps a com-
pact convex subset of a Banach space into itself has a fixed point".

For each fixed element $[z,v] \in C_{nm}[t_o,t_1]$, consider the control function $u(t)$ for $t \in [t_o,t_1]$

$$u(t) = B'(t)\Phi'(t_1,t;z,v)G^{-1}(t_o,t_1;z,v)[x_1 - \Phi(t_1,t_o;z,v)x_o$$
$$- \int_{t_o}^{t_1} \Phi(t_1,s;z,v)f(s,z,v)ds] \tag{6}$$

where $\Phi(t,t_o;z,v)$ is defined as that in Eq. (3). With this expression, Eq. (3) can be rewritten as

$$x(t) = \Phi(t,t_o;z,v)x_o$$

$$+ \int_{t_o}^{t} \Phi(t,s;z,v)B(s)B'(s)\Phi'(t_1,s;z,v)G^{-1}(t_o,t_1;z,v)ds \cdot [x_1-\Phi(t_1,t_o;z,v)x_o$$

$$- \int_{t_o}^{t_1} \Phi(t_1,s;z,v)f(s,z,v)ds] + \int_{t_o}^{t} \Phi(t,s;z,v)f(s;z,v)ds \tag{7}$$

It should be noted that by hypothesis (iii) $G^{-1}(t_o,t_1;z,v)$ is well-defined in above expression. It is easily seen that $x(t)$ in Eq. (7) satisfies both boundary conditions at $t = t_o$ and $t = t_1$.

Now the right sides of Eqs. (6)-(7) can be viewed as a pair of operators, $P_2([z,v])(t)$ and $P_1([z,v])(t)$, respectively. Define the nonlinear mapping

$$P([z,v])(t) = [P_1([z,v])(t), P_2([z,v])(t)] .$$

Obviously, $P_1([z,v])(t)$ and $P_2([z,v])(t)$ are continuous in t by the uniform continuity of $\Phi(t,t_o;z,v)$ in t. Therefore, P maps $C_{nm}[t_o,t_1]$ into $C_{nm}[t_o,t_1]$. It can also be easily verified that, by hopothesis (ii) and the definition of $\Phi(t,t_o;z,v)$, P is continuous in $[z,v]$.

Considering the subset of $C_{nm}[t_o, t_1]$

$$S = \left\{ [z,v] \in C_{nm}[t_o, t_1] : ||[z,v]|| \leqslant K , \right.$$

$$\left. ||[z(t), v(t)] - [z(\tilde{t}), v(\tilde{t})]|| \leqslant K|t-\tilde{t}|, \; \forall t, \tilde{t} \in [t_o, t_1] \right\}$$

where K is certain positive constant depending upon the bounds of A,B,f and their partial derivatives, it can be easily shown that the image set $P(S) \subset S$. Besides, S is closed and convex by this construction. Furthermore, each sequence $\{s_i\}_{i=1}^{\infty} \subset S$ constitutes a uniformly bounded equicontinuous family. Hence by the Arzela-Ascoli theorem [3], S is relative compact and therefore compact.

Then, the Schauder's theorem [3] can be applied to conclude that P has a fixed point $[z^*, v^*]$, i.e.,

$$P([z^*, v^*])(t) = [P_1([z^*, v^*]), P_2([z^*, v^*])] = [z^*, v^*]$$

Substitute this fixed point into Eqs. (6)-(7), a direct differentiation of Eq. (7) with respect to t shows that $z^*(t)$ is a solution to the system (1) for the control function $u(t)$ given by $v^*(t)$.

If condition (iii)(a) holds, $v^*(t)$ drives the system (1) from x_o to x_1 on some interval $[t_o, t_1]$ for all $x_o, x_1 \in R^n$, system (1) is globally completely controllable at t_o. If condition (iii)(b) holds, we have global total controllability.

Corollary 1: Given the system (1) with conditions (i)-(iii), then the perturbed system

$$\frac{d}{dt} x(t) = [A(t,x,u) + \epsilon \tilde{A}(t,x,u)]x(t) + [B(t) + \epsilon \tilde{B}(t)]u(t) + f(t,x,u), \qquad (8)$$

with $\tilde{A}, \tilde{B}$ satisfying the same type of conditions imposed on A,B, is globally controllable provided ϵ is sufficiently small.

Proof: One needs only to show that the determinant of the modified controllability matrix G* has a positive infimum. By observing that the determinant of $G^*(t_1, t_o; z, v, \varepsilon)$ can be expanded about $\varepsilon = 0$ [5] into the sum of det $G(t_1, t_o; z, v)$, $\varepsilon H(t_1, t_o; z, v)$ and $o(\varepsilon^2)$ where $H(t_1, t_o; z, v)$ is a bounded scalar, we know that by taking ε small enough there exists some $q^* > 0$ such that

$$\det G^*(t_1, t_o; z, v, \varepsilon) \geqslant q^* \quad \text{for all} \quad [z, v] \varepsilon C_{nm}[t_o, t_1] .$$

Therefore, Theorem 1 concludes the result of controllability.

Remarks: (i) If (A, f) do not explicitly depend on u(t), the theorem is applicable
to the class of systems with $B = B(t, x(t))$. These results then
reduce to the work reported in [4].

(ii) Uniform boundedness of partial derivatives can be slightly weakened
by considering a Lipschitz condition on corresponding variables.

III. LOCAL CONTROLLABILITY RESULT

Consider the following subset of $[t_o, t_1] \times R^n \times R^m$,

$$D = \{(t, x, u) : t_o \leqslant t \leqslant t_1, \quad |x| + |u| \leqslant d,$$
$$d \text{ is some positive constant}\} .$$

Theorem 2: The system (1) is locally (a) completely controllable at t_o, or
(b) totally controllable if the following conditions are satisfied.

(i) B(t) has a continuous first derivative with respect to t,

(ii) $A(t, x, u)$, $A_x(t, x, u)$, $A_u(t, x, u)$, $f(t, x, u)$ $f_x(t, x, u)$ and $f_u(t, x, u)$ are
continuous in D,

(iii) There exists a positive constant $\tilde{q}$ such that

$$\inf_{[z,v] \, \epsilon \, E} \det G(t_1,t_o;z,v) \geqslant \tilde{q}$$

where

$$E = \{[z,v] \, \epsilon \, C_{nm}[t_o,t_1] : ||[z,v]|| \leqslant d\}$$

and

$$M < \min \left\{\frac{d}{k_1} , \, d, \, \frac{d}{k_2}\right\}$$

where

M is the bound of f in D, k_1 and k_2 are certain positive constants

depending upon the bounds of A,B and their partial derivatives in D.

(a) for some $t_1 > t_o$, or (b) for all t_o and all $t_1 > t_o$.

Proof: The proof is similar to that of Theorem 1. We leave it to the
reader.

Corollary 2: Given the system (1) with conditions (i)-(iii) in Theorem 2,
the perturbed system

$$\frac{d}{dt} x(t) = [A(t,x,u) + \epsilon\tilde{A}(t,x,u)]x(t) + [B(t) + \epsilon\tilde{B}(t)]u(t) + f(t,x,u) \qquad (9)$$

with $\tilde{A},\tilde{B}$ satisfying the same type of conditions imposed upon A,B is locally
controllable provided ϵ is sufficiently small.

Proof: The proof is similar to that of Corollary 1 and follows immediately
from Theorem 2.

Example: Consider the system

$$\dot{x}_1 = x_2 + [\sin^2(x_1 + u)]x_2$$
$$\dot{x}_2 = x_1 + u + [\sin^2(x_1 + u)]x_1$$

In matrix form, we have

$$A(t,x,u) = \begin{bmatrix} 0 & 1 + \sin^2(x_1 + u) \\ 1 + \sin^2(x_1 + u) & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

It is easily seen that $B, dB/dt, A, \partial A/\partial x, \partial A/\partial u$ exist and are bounded in R and $R^2 \times R$, respectively. Furthermore, for each fixed $[z,v] \in C_{21}[t_o, t_1]$, the controllability matrix G is

$$G(t_o, t_1; z, v) = \frac{1}{4} \begin{bmatrix} \displaystyle\int_{t_o}^{t_1} (a-b)^2 dy & \displaystyle\int_{t_o}^{t_1} (a^2 - b^2) dy \\ \displaystyle\int_{t_o}^{t_1} (a^2 - b^2) dy & \displaystyle\int_{t_o}^{t_1} (a+b)^2 dy \end{bmatrix}$$

where

$$a = \exp \int_{t_o}^{t_1} [1 + \sin^2(z_1 + v)]dy, \quad b = \exp \int_{t_o}^{t_1} -[1 + \sin^2(z_1 + v)]dy .$$

It can be easily shown that $\det G(t_o, t_1; z, v) \geq q(t_o, t_1)$ for all $[z,v]$ where $q(t_o, t_1) = \frac{1}{16} [e^{2(t_1 - t_o)} - 1 - 2(t_1 - t_o) - 2(t_1 - t_o)^2 - \frac{4}{3}(t_1 - t_o)^3] > 0$ if $t_1 > t_o$. Hence, by Theorem 1 the system is globally totally controllable. However, the result of [1] cannot be applied here because in this case the non-linearity $f(t,x,u) = [x_2 \sin^2(s_1 + u), x_1 \sin(x_1 + u)]'$ is not bounded in $R \times R^2 \times R$.

IV. CONCLUSION

Sufficient conditions for global and local controllability of a class of nonlinear systems have been given. A natural question arises as to whether or not these results hold by the same fixed point arguments if the system matrices (A,f) are parameterized as $A(t,p(t),q(t))$ and $f(t,p(t),q(t))$ where $[p,q] \in C_{nm}[t_o,t_1]$ so that for each given $[p,q]$ the parametrized linear time-varying system is controllable. The answer is negative because boundedness and continuity of the partial derivatives of relevant matrices are required in addition to the nonsingularness of the controllability matrix.

Acknowledgment

The author wishes to express his gratitude to Professor A. E. Pearson for his encouragement and helpful comments to this work.

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3/76

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AFOSR TR-76-1153		

4. TITLE (and Subtitle)	5. TYPE OF REPORT & PERIOD COVERED
A CLASS OF CONTROLLABLE NONLINEAR SYSTEMS	Interim rept.
	6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)	8. CONTRACT OR GRANT NUMBER(s)
Kuang-Chung Wei	AFOSR 75-2793

9. PERFORMING ORGANIZATION NAME AND ADDRESS	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
Brown University Division of Engineering Providence, Rhode, Island 02912	61102F 9769-01

11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE
Air Force Office of Scientific Research (NM) Bolling AFB, DC 20332	1976
	13. NUMBER OF PAGES
	9

14. MONITORING AGENCY NAME & ADDRESS(If different from Controlling Office)	15. SECURITY CLASS. (of this report)
AF-AFOSR-2793-75	UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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